

# Understanding the $\lambda$ -calculus via (non-)linearity and rewriting

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# Outline

- 1 Introduction: lambda-calculi and a unifying meta-theory
- 2 Girard's two translations: from proof theory to computation
- 3 The bang calculus: its syntax and semantics
- 4 Embedding CbN and CbV  $\lambda$ -calculi into the bang calculus, syntactically
- 5 Embedding CBN and CBV  $\lambda$ -calculi into the bang calculus, semantically
- 6 Conclusions

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## From many $\lambda$ -calculi ...

The  $\lambda$ -calculus is the model of computation underlying

- functional programming languages (Haskell, OCaml, ...)
- proof assistants (Coq, Isabelle/Hol, ...).

Actually, there are **many** lambda-calculi, depending on

- the evaluation mechanism (e.g., call-by-name, call-by-value, call-by-need);
- computational feature the calculus aims to model (e.g., pure, non-deterministic);
- the type system (e.g. untyped, simply typed, second order).

The existence of  $N$  **separate paradigms** is troubling:

- it makes each calculus appear arbitrary (is there a more canonical language?)
- each time we create a new style of semantics (e.g. operational, denotational, continuations, etc.) we always need to do it  $N$  times—once for each paradigm.

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**Goal:** A unifying and robust meta-theory of  $\lambda$ -calculi, rooted on:

- 1 **linear logic**  $\rightsquigarrow$  unifying setting for different evaluation mechanisms;
- 2 **elementary rewriting**  $\rightsquigarrow$  modular and robust approach to extensions of  $\lambda$ -calculi.

In this talk: focus on Point 1.

- the role of linear logic;
- different evaluation mechanisms.

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## The role of linear logic with respect to $\lambda$ -calculi

Girard's linear logic (1987) provides new concepts and tools to **study**  $\lambda$ -calculi:

- 1 denotational models of linear logic provides denotational models for  $\lambda$ -calculi;
- 2 clear notion of resource and linear consumption  
 $f: A \multimap B \approx f$  consumes a value of type  $A$  and transforms it into a value of type  $B$ ;
- 3 quantitative analysis of computation
  - ▶ semantic tools to study execution time (De Carvalho *et al.*);
  - ▶ "compatible" with cost models (Accattoli *et al.*).
- 4 ...

LL also hints how to **modify** syntax and dynamics of  $\lambda$ -calculi to have "good properties".

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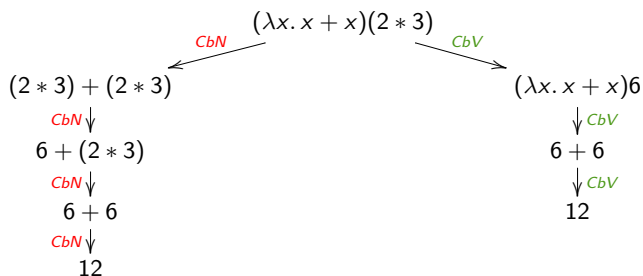
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## Call-by-Name vs. Call-by-Value (for dummies)

- Call-by-Name (**CbN**): pass the argument to the calling function **before** evaluating it.
- Call-by-Value (**CbV**): pass the argument to the calling function **after** evaluating it.

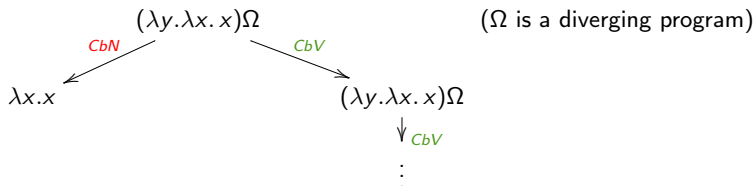
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Summing up, **CbV** is **eager**, that is,

- 1 **CbV** is **smarter** than **CbN** when the argument must be duplicated;
- 2 **CbV** is **sillier** than **CbN** when the argument must be discarded.



## The CbN and CbV $\lambda$ -calculi

$\lambda$ -terms:	$t, s, r ::= v \mid ts$	(set: $\Lambda$ )
$\lambda$ -values:	$v ::= x \mid \lambda x t$	(set: $\Lambda_v$ )

*Reductions:*  $(\lambda x t)s \rightarrow_{\beta} t\{s/x\}$  (CbN)       $(\lambda x t)v \rightarrow_{\beta_v} t\{v/x\}$  (CbV)

CbN and CbV  $\lambda$ -calculi have different operational and denotational semantics  
 $\rightsquigarrow$  in general, it is impossible to derive a property for CbV from CbN, or vice versa.

Examples, with  $I := \lambda z.z$  (identity) and  $\delta := \lambda z.zz$  (duplicator):

- 1  $(\lambda y.I)(\delta\delta)$   $\beta$ -normalizes but  $\beta_v$ -diverges

$$(\lambda y.I)(\delta\delta) \rightarrow_{\beta} I \quad (\lambda y.I)(\delta\delta) \rightarrow_{\beta_v} (\lambda y.I)(\delta\delta) \rightarrow_{\beta_v} \dots$$

- 2  $(\lambda x.\delta)(xx)\delta$  is  $\beta_v$ -normal but  $\beta$ -divergent:  $(\lambda x.\delta)(xx)\delta \rightarrow_{\beta} \delta\delta \rightarrow_{\beta} \delta\delta \rightarrow_{\beta} \dots$

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Logic		Computer Science
formula	$\longleftrightarrow$	type
proof	$\longleftrightarrow$	program
cut-elimination	$\longleftrightarrow$	evaluation
coherence	$\longleftrightarrow$	termination
different encodings of intuitionistic arrow in LL	$\longleftrightarrow$	different evaluation mechanisms

↪ Tools from intuitionistic linear logic (ILL) can be used to study properties of:

- call-by-name evaluation via Girard's translation  $(\cdot)^N$ ,
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## The two Girard's translations of IL into ILL (1987)

well-known translation  $(\cdot)^N$

$$X^N = X$$

$$(A \rightarrow B)^N = !A^N \multimap B^N$$

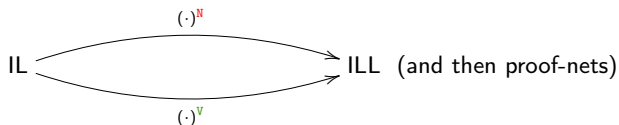
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"boring" translation  $(\cdot)^V$

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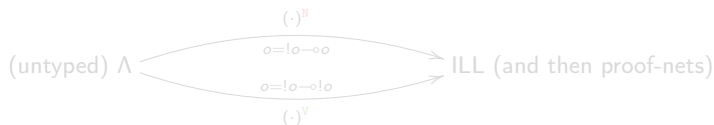
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simply typed  $\lambda = \text{IL}$  (via Curry-Howard)

(untyped)  $\lambda = \text{IL} + \text{unique atomic type } o + \text{type identity } o = o \rightarrow o$



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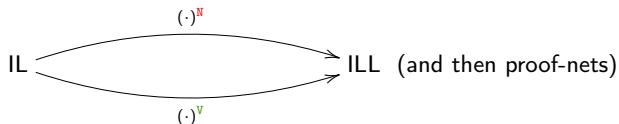
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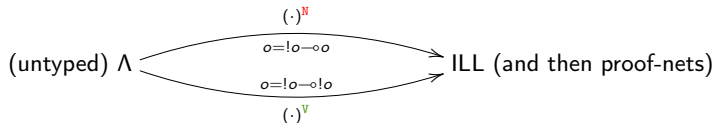
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# Girard's first translation: $(\cdot)^N$

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 X^N &= X \\
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(natural deduction for IL)

(sequent calculus for ILL)

$$\frac{}{x:A \vdash x:A} \text{ax} \rightsquigarrow \frac{}{A^N \vdash A^N} \text{ax} \rightsquigarrow \frac{}{!A^N \vdash A^N} \text{der}$$

$$\frac{\Gamma, x:A \vdash M:B}{\Gamma \vdash \lambda x.M : A \rightarrow B} \rightarrow_i \rightsquigarrow \frac{! \Gamma^N, !A^N \vdash B^N}{! \Gamma^N \vdash !A^N \multimap B^N} \multimap$$

$$\frac{\Gamma \vdash M:A \rightarrow B \quad \Delta \vdash N:A}{\Gamma, \Delta \vdash MN:B} \rightarrow_e \rightsquigarrow \frac{\frac{! \Delta^N \vdash A^N}{! \Delta^N \vdash !A^N} ! \quad \frac{}{B^N \vdash B^N} \text{ax}}{! \Delta^N, !A^N \multimap B^N \vdash B^N} \otimes \rightsquigarrow \frac{}{! \Gamma^N \vdash !A^N \multimap B^N} \text{cut}$$

The translation  $(\cdot)^N$  puts a ! in front of every formula on the left-hand side of  $\vdash$   
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## Girard's second ("boring") translation: $(\cdot)^V$

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 (\Gamma \vdash A)^V &= !\Gamma^V \vdash !A^V
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(natural deduction for IL)

(sequent calculus for ILL)

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$$\frac{!\Gamma^V, !A^V \vdash !B^V}{!\Gamma^V \vdash !A^V \multimap !B^V} \multimap \rightsquigarrow$$

$$\frac{}{!\Gamma^V \vdash !(A^V \multimap B^V)} !$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Delta \vdash N : A}{\Gamma, \Delta \vdash MN : B} \rightarrow_e \rightsquigarrow$$

$$\frac{\frac{!\Delta^V \vdash !A^V \quad !B^V \vdash !B^V}{!\Delta^V, !A^V \multimap !B^V \vdash !B^V} \otimes}{!\Gamma^V \vdash !(A^V \multimap B^V) \quad !\Delta^V, !(A^V \multimap B^V) \vdash !B^V} \text{ der} \rightsquigarrow$$

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An example: from IL (natural deduction)...

$$\pi = \frac{\frac{\frac{\overline{a:A \vdash a:A}^{ax}}{a:A, c:C \vdash a:A}^w}{a:A \vdash \lambda c a. C \rightarrow A} \rightarrow_i \quad \frac{\frac{\overline{x:B \rightarrow C \vdash x:B \rightarrow C}^{ax} \quad \frac{\overline{b:B \vdash b:B}^{ax}}{b:B, x:B \rightarrow C \vdash xb. C} \rightarrow_e}{b:B, x:B \rightarrow C \vdash xb. C}}{a:A, b:B, x:B \rightarrow C \vdash (\lambda c a)(xb). A} \rightarrow_e}{a:A, b:B, x:B \rightarrow C \vdash (\lambda c a)(xb). A} \rightarrow_e$$

↓cut

$$\text{nf}(\pi) = \frac{\frac{\overline{a:A \vdash a:A}^{ax}}{a:A, b:B, \vdash a:A}^w}{a:A, b:B, x:B \rightarrow C \vdash a:A}^w$$



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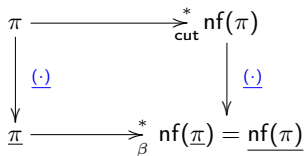
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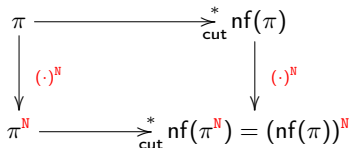
An example: ... to ILL via  $(\cdot)^N$

intuit. logic (IL):

$(\cdot)^N$



intuit. LL (ILL):



An example: ... to ILL via  $(\cdot)^N$

$$\pi^N = \frac{\frac{\frac{\frac{\overline{A \vdash A}^{ax}}{!A \vdash A}^{der}}{!A, !C \vdash A}^w}{!A \vdash !C \multimap A}^{\wp}}{\frac{\frac{\frac{\frac{\overline{!B \multimap C \vdash !B \multimap C}^{ax}}{!(B \multimap C) \vdash !B \multimap C}^{der}}{!B, !(B \multimap C) \vdash C}^{\otimes}}{!B, !(B \multimap C) \vdash !C}^{\otimes}}{!C \multimap A, !B, !(B \multimap C) \vdash A}^{cut}}{A \vdash A}^{ax}}{!C \multimap A, !B, !(B \multimap C) \vdash A}^{cut}}{a:!A, b:!B, x:!(B \multimap C) \vdash (\lambda c a)(xb):A}^{cut}}$$

cut  $\downarrow_+$

$$\text{nf}(\pi^N) = \frac{\frac{\frac{\overline{a:A \vdash a:A}^{ax}}{a:!A \vdash a:A}^{der}}{a:!A, b:!B, \vdash a:A}^w}{a:!A, b:!B, x:!(B \multimap C) \vdash a:A}^w = (\text{nf}(\pi))^N$$

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An example: ... to ILL via  $(\cdot)^V$

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$(\cdot)^V$   
↓

intuit. LL (ILL):

$$\begin{array}{ccc} \pi & \xrightarrow{\text{cut}^*} & \text{nf}(\pi) \\ \downarrow (\cdot)^V & & \downarrow (\cdot)^V \\ \pi^V & \xrightarrow{\text{cut}^*} & \text{nf}(\pi^V) \neq (\text{nf}(\pi))^V \end{array}$$





In the  $\lambda$ -calculus there are two evaluation mechanisms:

- *call-by-name* (CbN,  $\beta$ -reduction): no restriction in firing a  $\beta$ -redex;
- *call-by-value* (CbV,  $\beta_v$ -reduction): a  $\beta$ -redex  $(\lambda x t)s$  can be fired only if  $s$  is a **value**.

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- via  $(\cdot)^N$  every argument is translated by a box  
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- via  $(\cdot)^V$  every (and only) abstraction or variable is translated by a box  
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Yes! There are several examples in the literature, *e.g.*



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We propose here an alternative solution, the **bang calculus**: it differs from the linear  $\lambda$ -calculi of Maraist *et al.* for some technical aspects.

All the “good” results proved for Maraist *et al.*'s linear  $\lambda$ -calculus hold also for the bang calculus, and in a “better” way.

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# Syntax and reduction rules of the bang calculus

## Syntax:

*Terms*  $T, S, R ::= x \mid T^! \mid \lambda x T \mid TS \mid \text{der } T$  (set:  $!\wedge$ )

## Reduction rules:

$(\lambda x T)S^! \mapsto_\lambda T\{S/x\}$        $\text{der}(T^!) \mapsto_d T$        $\mapsto_b := \mapsto_\lambda \cup \mapsto_d$

For any  $r \in \{\lambda, d, b\}$ ,

- *r-reduction*  $\rightarrow_r$  is the closure under any contexts of  $\mapsto_r$  (it fires any *r*-redex);
- *ground r-reduction* (or *r<sub>g</sub>-reduction*)  $\rightarrow_{r_g}$  does not fire *r*-redexes under  $!$ .

*Idea*: only boxes  $T^!$  are duplicable and discardable ( $\rightsquigarrow$  linear logic).

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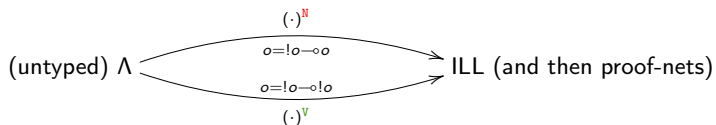
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The bang calculus gives a nice decomposition of the two Girard's translations  $(\cdot)^N$  and  $(\cdot)^V$ .

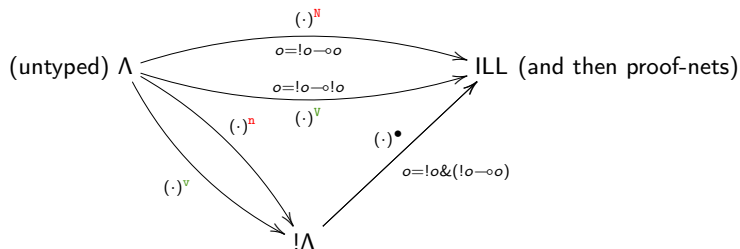


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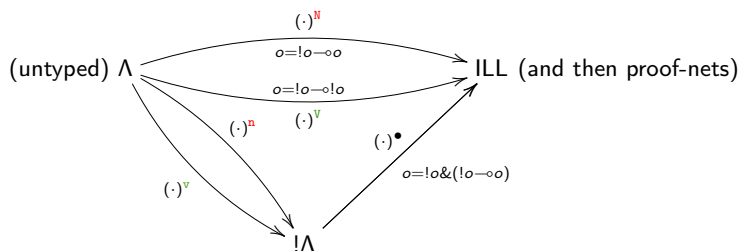
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## Denotational models for LL

A *denotational model for LL* is given by:

- 1 a symmetric monoidal closed category  $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \sigma)$ ; we use  $X \multimap Y$  for the object of linear morphisms from  $X$  to  $Y$ ;
- 2  $\mathcal{L}$  is cartesian with terminal object  $\top$  and product  $\&$ ;  
 $\mathcal{L}$  is cocartesian with initial object  $0$  and coproduct  $\oplus$ ;
- 3 a comonad  $!_{-} : \mathcal{L} \rightarrow \mathcal{L}$  with counit  $\text{der}_X \in \mathcal{L}(!X, X)$  (*dereliction*) and comultiplication  $\text{dig}_X \in \mathcal{L}(!X, !!X)$  (*digging*);
- 4 ...

**Specific assumption:** the unique morphism in  $\mathcal{L}(0, \top)$  is an iso; to simplify,  $0 = \top$ .  
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## Denotational models for the bang calculus built from LL

To have a model of the untyped bang calculus, take a **retraction**: an object  $\mathcal{U}$  such that

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**Rmk (retractions):** It follows that  $!\mathcal{U} \triangleleft \mathcal{U}$  and

- 1  $!\mathcal{U} \multimap \mathcal{U} \triangleleft \mathcal{U}$  (semantic **CbN** version of  $o \rightarrow o = o$ );
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A term  $T$  with  $\vec{x} = (x_1, \dots, x_n) \supseteq \text{fv}(T)$  is interpreted by a morphism  $\llbracket T \rrbracket_{\vec{x}}: (!\mathcal{U})^{\otimes n} \rightarrow \mathcal{U}$ . The (omitted) definition is by induction on  $T$ , using the morphisms in  $\mathcal{L}$  sketched before.

**Theorem (invariance under reduction; G. & Ehrhard 2016)**

Let  $T, S \in !\Lambda$  and  $\text{fv}(T) \subseteq \vec{x}$ . If  $T \rightarrow_b S$  then  $\llbracket T \rrbracket_{\vec{x}} = \llbracket S \rrbracket_{\vec{x}}$ .



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**CbN** translation  $(\cdot)^n: \Lambda \rightarrow !\Lambda$

$$x^n = x$$

$$(\lambda x t)^n = \lambda x t^n$$

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**CbV** translation  $(\cdot)^v: \Lambda \rightarrow !\Lambda$

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The difference between  $(\cdot)^n$  and  $(\cdot)^v$  is only where  $(\cdot)^!$  and  $\text{der}$  are placed.

**Idea:** in **CbN** any argument is duplicable and discardable, in **CbV** only  $\lambda$ 's and variables.

Lemma (translations commute with substitution)

Let  $t$  and  $s$  be  $\lambda$ -terms.

- 1 One has that  $t^n\{s^n/x\} = (t\{s/x\})^n$ .
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# The bang calculus is a conservative extension of CbN and CbV $\lambda$ -calculi

## Theorem (Sound and complete simulations; G. & Manzonetto, 2018)

Let  $t$  be a  $\lambda$ -term.

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*Soundness:* If  $t \rightarrow_{\beta} t'$  then  $t^n \rightarrow_{\lambda} t'^n$  (and  $t^n \rightarrow_b t'^n$ );

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These simulations are:

- **modular:** the ground CbX  $\lambda$ -calculus is simulated in the ground bang calculus (ground CbN = head reduction, ground CbV = weak reduction);
- **quantitative sensitive:** one  $\beta$ -step is simulated by one  $\lambda$ -step, and conversely.

In other linear  $\lambda$ -calculi, completeness of CBV translation fails!

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*Soundness:* If  $t \rightarrow_{\beta^v} t'$  then  $t^v \rightarrow_d \rightarrow_{\lambda} t'^v$  (and hence  $t^v \rightarrow_b \rightarrow_b t'^v$ );

*Completeness:* If  $t^v \rightarrow_d \rightarrow_{\lambda} S$  then  $S = t'^v$  and  $t \rightarrow_{\beta^v} t'$  for some  $\lambda$ -term  $t'$ .

These simulations are:

- **modular:** the ground CbX  $\lambda$ -calculus is simulated in the ground bang calculus (ground CbN = head reduction, ground CbV = weak reduction);
- **quantitative sensitive:** one  $\beta$ -step is simulated by one  $\lambda$ -step, and conversely.

In other linear  $\lambda$ -calculi, completeness of CBV translation fails!

~ A step in the CbV fragment of those calculi need not correspond to a  $\beta^v$ -step.

# The bang calculus is a conservative extension of CbN and CbV $\lambda$ -calculi

## Theorem (Sound and complete simulations; G. & Manzonetto, 2018)

Let  $t$  be a  $\lambda$ -term.

### 1 Conservative extension of CbN $\lambda$ -calculus:

*Soundness:* If  $t \rightarrow_{\beta} t'$  then  $t^n \rightarrow_{\lambda} t'^n$  (and  $t^n \rightarrow_b t'^n$ );

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These simulations are:

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## The target of the CbN translation in the bang calculus

target of **CbN** translation into  $!\Lambda$ :  $T, S ::= x \mid TS^! \mid \lambda x T$  (set:  $!\Lambda^n$ )

**Rmk:**  $t^n \in !\Lambda^n$  for any  $t \in \Lambda$ , and conversely, for any  $T \in !\Lambda^n$ ,  $T^n = t$  for some  $t \in \Lambda$ .

**Rmk:** In  $!\Lambda^n$ , the construct  $\text{der}$  never occurs  $\rightsquigarrow$  in  $!\Lambda^n$ , hence  $\rightarrow_\lambda = \rightarrow_b$ .

$\rightsquigarrow$  The CbN  $\lambda$ -calculus is **isomorphic** to the fragment  $!\Lambda^n$  of the bang calculus.

$$\begin{array}{ccc} \Lambda \ni t & \xrightarrow{\beta} & s \in \Lambda \\ \downarrow (\cdot)^n & & \downarrow (\cdot)^n \\ !\Lambda \ni t^n & \xrightarrow{b} & s^n \in !\Lambda \end{array}$$

Corollary (Preservations with respect to CbN  $\lambda$ -calculus; G. & Manzonetto, 2018)

- 1 (CbN equational theory) Let  $t, s \in \Lambda$ :  $t \simeq_\beta s$  iff  $t^n \simeq_b s^n$ .
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## The target of the CbV translation in the bang calculus

target of *CbV* translation into  $!\Lambda$ :  $M, N ::= U^! \mid \text{der } MN \mid UM$  (set:  $!\Lambda^v$ )  
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**Rmk:** For any  $t \in \Lambda$ ,  $t^v \in !\Lambda^v$ ; in particular, for any  $v \in \Lambda_v$ ,  $v^v = U^!$  for some  $U \in !\Lambda^v$ .

The converse fails:  $(\lambda x xx)^v = \Delta' \rightarrow_d \Delta$ , where  $\Delta$  is b-normal and  $\nexists$   $\lambda$ -term  $t : t^v = \Delta$ .  
 Note that  $\lambda x xx$  is  $\beta^v$ -normal but  $(\lambda x xx)^v = \Delta'$  is not b-normal.

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Corollary (Preservations with respect to CbV  $\lambda$ -calculus; G & Manzonetto, 2018)

① (*CbV equational theory*) Let  $t, s \in \Lambda$ :  $t \simeq_{\beta^v} s$  iff  $t^v \simeq_{\text{b}} s^v$ .

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## Factorization of the semantics of CBN $\lambda$ -calculus (G. & Manzonetto, 2018)

Retraction  $!U \multimap U \triangleleft U$  in  $\mathcal{L}$  lifts to a retraction  $!U \multimap U \triangleleft U$  in  $\mathcal{L}_!$  (co-Kleisli of  $\mathcal{L}$  via !)  
 $\rightsquigarrow$  any denotational model  $\mathcal{U}$  of the bang calculus is a denotat. model of **CbN**  $\lambda$ -calculus.

$[t]_{\mathcal{X}}^n =$  usual **CbN** interpretation of a  $\lambda$ -term  $t$  in  $\mathcal{U}$ .

$\llbracket t^n \rrbracket_{\mathcal{X}} =$  bang interpretation of the **CbN** translation of a  $\lambda$ -term  $t$  in  $\mathcal{U}$ .

**Question:** For a  $\lambda$ -term  $t$ , what is the relationship between  $[t]^n$  and  $\llbracket t^n \rrbracket$ ?

**Theorem (Factorization of any denotational semantics of CBN  $\lambda$ -calculus)**

For every  $\lambda$ -term  $t$ ,  $[t]_{\mathcal{X}}^n = \llbracket t^n \rrbracket_{\mathcal{X}}$  (up to Seely's isos).

$$\begin{array}{ccccc} \Lambda \ni t & \xrightarrow{\quad} & [\cdot]^n & \xrightarrow{\quad} & [t]^n = \llbracket t^n \rrbracket \in \mathcal{U} \\ & \searrow & \circlearrowleft & \nearrow & \\ & (\cdot)^n & & [\cdot] & \\ & & t^n \in !\Lambda & & \end{array}$$



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## Factorization of the semantics of CbV $\lambda$ -calculus? (G. & Manzonetto, 2018)

The functor  $!$  is a strong monad on the Kleisli category  $\mathcal{L}_!$   $\rightsquigarrow$  retraction  $!U \dashv \dashv !U \triangleleft U$   
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Theorem (Non-factorization of *some* denotational semantics of CbV  $\lambda$ -calculus)

In the relational semantics, there is a  $\lambda$ -term  $t$  such that  $[t]_{\bar{x}}^v \subsetneq \llbracket t^v \rrbracket_{\bar{x}}$ .

$$\begin{array}{ccc} \Lambda \ni t & \xrightarrow{[\cdot]^v} & [t]^v \neq \llbracket t^v \rrbracket \in \mathcal{U} \\ & \searrow (\cdot)^v & \nearrow [\cdot] \\ & t^v \in !\Lambda & \end{array}$$

Conjecture: There still exists a relationship in CbV between  $[t]^v$  and  $\llbracket t^v \rrbracket$ , but it should be more sophisticated than in CbN. Maybe we should use logical relations between the two.

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## Conclusions (and motivations)

- 1 The existence of **two separate paradigms** (**CbN** and **CbV**  $\lambda$ -calculi) is troubling:
  - ▶ it makes each language appear arbitrary (a unified language might be more canonical);
  - ▶ each time we create a new style of semantics (e.g. operational semantics, continuation semantics, Scott semantics, game semantics, etc.) we always need to do it twice.
- 2 The bang calculus is a general setting to compare **CbN** and **CbV**  $\lambda$ -calculi in the **same** rewriting system and with the **same** denotational semantics.
  - ▶ **CbN**  $\lambda$ -calculus has a rich and refined theory featuring advanced concepts such as separability, solvability, Böhm trees, classification of  $\lambda$ -theories, full-abstraction, etc.
  - ▶ This is not the case for **CbV**  $\lambda$ -calculus: in the **CbV** counterpart of these theoretical notions there are only partial and not satisfactory results (or do not exist at all!).
  - ▶ These theoretical notions and results well studied for the **CbN**  $\lambda$ -calculus might be adapted and studied in the more general setting of the bang calculus  $\rightarrow$  compelling point of view to analyze the corresponding notions for **CbV**  $\lambda$ -calculus.

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- ① **Factorization theorem** proved once and for all in the bang calculus:

$$T \rightarrow_b^* S \implies T \rightarrow_{b_g}^* \rightarrow_{\neg b_g}^* S$$

↪ By translation in **CbN** and **CbV**  $\lambda$ -calculi:

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(Faggian, G., FoSSaCS 2021)

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$$\llbracket T \rrbracket_{\bar{x}} \text{ is non-trivial} \iff T \text{ is } \mathbf{b_g}\text{-normalizable}$$

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↪ bi-categorical setting for a **proof-relevant semantics**.

$$\llbracket T \rrbracket_{\bar{x}} = \left\{ \tilde{\pi} \mid \begin{array}{c} \vdots \\ \pi \\ \Gamma \vdash T : a \end{array} \right\}$$

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In non-idempotent intersection type systems for  $\lambda$ -calculi, typability is undecidable.

**Question:** Is the **inabitation** problem decidable in the bang calculus?

Given an typing context  $\Gamma$  and a multi type  $M$ , is there a term  $t$  such that

$$\Gamma \vdash t : M \text{ is derivable?}$$

**Question bis:** Same question, but in the **CbN** and **CbV**  $\lambda$ -calculi.

**Answer** [ArrKesGue23]: Yes, it is decidable and we can find all the inhabitants!

And if we restrict the search space of our algorithm to:

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In non-idempotent intersection type systems for  $\lambda$ -calculi, typability is undecidable.

**Question:** Is the **inabitation** problem decidable in the bang calculus?

Given an typing context  $\Gamma$  and a multi type  $M$ , is there a term  $t$  such that

$$\Gamma \vdash t : M \text{ is derivable?}$$

**Question bis:** Same question, but in the **CbN** and **CbV**  $\lambda$ -calculi.

**Answer** [ArrKesGue23]: Yes, it is decidable and we can find all the inhabitants!

And if we restrict the search space of our algorithm to:

- the **CbN** fragment of the bang calculus  $\rightsquigarrow$  we decide inhabitation in **CbN**;
- the **CbV** fragment of the bang calculus  $\rightsquigarrow$  we decide inhabitation in **CbV**.

## The big open question

Call-by-Need is another evaluation mechanism (e.g., used by Haskell):

- as smart as CbV for duplication,
- as smart as CbN for erasure.

Question: What about Call-by-Need? Can we use LL to understand Call-by-Need?

Question bis: Is there a general framework subsuming CbV, CbN and CbNeed?

Idea: We should split the ! comonad into two:

- one for duplication;
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Thank you!

Questions?

